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LETTER TO THE EDITOR

Solution to the spherical Raman–Nath equation

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Abstract. The spherical Raman-Nath equation, which describes the stimulated Compton scattering, is transformed into a Fokker-Planck type equation for 'diffusion' over the spherical surface. This is done by using the Q representation in the phase space of the atomic coherent states. With the quantum electron recoil as the perturbation parameter, a perturbative solution is obtained up to arbitrarily high order, hence approaching an exact solution.

The stimulated Compton scattering (SCS) is of great current interest because it is the fundamental process in a free-electron laser. Dattoli and Renieri (1984) have shown that the SCS can be described by the following difference-differential equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_n(t) = (-2n\delta + n^2\varepsilon)C_n(t) + \lambda[(N-n)(n+1)]^{1/2}C_{n+1}(t) + \lambda[(N-n+1)n]^{1/2}C_{n-1}(t) \quad (1)$$

where $C_n(t)$ is the probability amplitude that n photons are propagating forward at time t, N is a constant integer, λ is the coupling constant, and δ and ε are parameters related to the initial electron momentum and the quantum electron recoil respectively.

Equation (1) has been recognised as a generalised Raman-Nath (RN) equation by Bosco *et al* (1984). Hence these authors call it the spherical RN equation. The original RN equation was derived to describe light diffraction by ultrasound (Raman and Nath 1937). The RN type equations are important because they appear in a large number of physical phenomena, as pointed out by Bosco and Dattoli (1983).

Bosco *et al* (1984) have obtained an exact solution to equation (1) under the initial condition

$$C_n(0) = \delta_{n,0} \quad (2)$$

and under the simplifying assumption that $\varepsilon = 0$. The $n^2\varepsilon$ term, because of its quadratic dependence on n, is the main obstacle in solving all RN type equations. Recently, the present author (Lee 1985) has obtained a perturbative solution of equation (1) to the first order of ε. In this letter, we try to push the perturbative solution to arbitrarily high order of ε, approaching an exact solution.

It has been mentioned by Bosco *et al* (1984) that equation (1) can be viewed as describing the time evolution of the atomic coherents (ACS) (Arecchi *et al* 1972) driven by external fields. So it is quite natural to use ACS as the basis for constructing the solution to equation (1). We specifically choose the Q representation of ACS in the phase space for our purpose.

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The density matrix to be constructed from the solution of equation (1) are of the following form:

$$\rho(t)=\sum_{n=0}^{N}\sum_{m=0}^{N}C_m^*(t)C_n(t)|n\rangle\langle m| \tag{3}$$

where the $|n\rangle$ are photon number states. The ACS are defined as

$$|\theta,\phi\rangle=\sum_{n=0}^{N}|n\rangle\binom{N}{n}^{1/2}(\cos\theta/2)^{N-n}(\sin\theta/2)^n\,\mathrm{e}^{-\mathrm{i}n\phi} \tag{4}$$

and the probability distribution over the spherical surface in the Q representation is defined as

$$\begin{aligned}Q(\theta,\phi,t)&\equiv\frac{N+1}{4\pi}\langle\theta,\phi|\rho(t)|\theta,\phi\rangle\\ &=\frac{N+1}{4\pi}\sum_{m=0}^{N}\sum_{n=0}^{N}\binom{N}{m}^{1/2}\binom{N}{n}^{1/2}C_m^*(t)C_n(t)\\ &\quad\times(\cos\theta/2)^{2N-m-n}(\sin\theta/2)^{m+n}\,\mathrm{e}^{-\mathrm{i}(m-n)\phi}\end{aligned} \tag{5}$$

where we have used equations (3) and (4) and the orthonormality of the $|n\rangle$. By the definition of equation (5) we see that $Q(\theta,\phi,t)$ is non-negative. It is the phase space distribution function corresponding to the antinormal ordering of the creation and annihilation operators. Similar representation has been used to describe superfluorescence (Lee 1984).

Using equation (1) in equation (5), we obtain the following partial differential equation

$$\begin{aligned}\{\partial/\partial t&+2\lambda\sin\phi\,\partial/\partial\theta+(-2\delta+2\lambda\cot\theta\cos\phi)\,\partial/\partial\phi\\ &+\varepsilon[\sin\theta\,\partial/\partial\theta+N(1-\cos\theta)]\,\partial/\partial\phi\}Q(\theta,\phi,t)=0\end{aligned} \tag{6}$$

which is of the Fokker-Planck type.

We look for a perturbative solution to equation (6) with ε as the perturbation parameter and under the initial condition

$$Q(\theta,\phi,0)=\frac{N+1}{4\pi}[(1+\cos\theta)/2]^N \tag{7}$$

obtained by using equation (2) in equation (5). Let

$$\hat{D}_0\equiv\partial/\partial t+2\lambda\sin\phi\,\partial/\partial\theta+(-2\delta+2\lambda\cot\theta\cos\phi)\,\partial/\partial\phi \tag{8}$$

$$\hat{D}_1\equiv N(1-\cos\theta)\,\partial/\partial\phi+\sin\theta\,\partial^2/\partial\theta\,\partial\phi \tag{9}$$

and

$$Q(\theta,\phi,t)=Q_0(\theta,\phi,t)+\varepsilon Q_1(\theta,\phi,t)+\varepsilon^2Q_2(\theta,\phi,t)+\ldots. \tag{10}$$

Then substitution of equations (8)-(10) into equation (6) yields the recurrence relation

$$\hat{D}_0Q_{l+1}(\theta,\phi,t)+\hat{D}_1Q_l(\theta,\phi,t)=0. \tag{11}$$

As long as $l\ll N$, the solution to equation (11) can be written in the form

$$Q_l(\theta,\phi,t)=(2N)^l\binom{N/2}{l}[F(\theta,\phi,t)]^{N-2l}[G(\theta,\phi,t)]^l \tag{12}$$

where the F and G must satisfy the following equations:

$$\hat{D}_0 F = 0 \tag{13a}$$

and

$$\hat{D}_0 G = -[(1-\cos\theta)F + \sin\theta\,\partial F/\partial\theta]\,\partial F/\partial\phi. \tag{13b}$$

They can be expressed in terms of the spherical harmonics up to the first order and the second order, respectively, as follows:

$$F = f_0 + f_1(t)\cos\theta + f_2(t)\sin\theta\cos\phi + f_3(t)\sin\theta\sin\phi \tag{14a}$$

and

$$G = g_1(t)\cos\theta + g_2(t)\sin\theta\cos\phi + g_3(t)\sin\theta\sin\phi + g_4(t)(3\cos^2\theta - 1) + g_5(t)\sin\theta\cos\theta\cos\phi + g_6(t)\sin\theta\cos\theta\sin\phi + g_7(t)\sin^2\theta\cos 2\phi + g_8(t)\sin^2\theta\sin 2\phi. \tag{14b}$$

The coefficients in equation (14a) can be obtained easily by solving equation (13a) with the initial condition equation (7) as follows:

$$f_0 = \tfrac{1}{2} \tag{15a}$$

$$f_1 = \tfrac{1}{2}[(\delta/\omega)^2 + (\lambda/\omega)^2\cos 2\omega t] \tag{15b}$$

$$f_2 = (\lambda\delta/2\omega^2)[1-\cos 2\omega t] \tag{15c}$$

and

$$f_3 = (\lambda/2\omega)\sin 2\omega t \tag{15d}$$

where we have adopted the notation

$$\omega \equiv (\lambda^2 + \delta^2)^{1/2}. \tag{16}$$

Substituting equations (15a)-(15d) into equation (14a) and using it in equation (13b), we can then solve for G to obtain the coefficients in equation (14b) as follows:

$$g_1 = -(\lambda^4\delta/32\omega^6)(9 - 8\cos 2\omega t - \cos 4\omega t - 12\omega t\sin 2\omega t) \tag{17a}$$

$$g_2 = (\lambda/\delta)g_1 + (\lambda^3/16\omega^4)(3 - 2\cos 2\omega t - \cos 4\omega t - 6\omega t\sin 2\omega t) \tag{17b}$$

$$g_3 = (\lambda^3\delta/16\omega^5)(\sin 2\omega t + \sin 4\omega t - 6\omega t\cos 2\omega t) \tag{17c}$$

$$g_4 = (\lambda^4\delta/64\omega^6)[3 - 8\cos 2\omega t + 5\cos 4\omega t + 4\omega t(\sin 2\omega t + \sin 4\omega t)] \tag{17d}$$

$$g_5 = (4\lambda/\delta)g_4 - (\lambda^3/8\omega^4)[1 - 3\cos 2\omega t + 2\cos 4\omega t + \omega t(\sin 2\omega t + 2\sin 4\omega t)] \tag{17e}$$

$$g_6 = -(\lambda^3\delta/16\omega^5)[5\sin 2\omega t - 4\sin 4\omega t + 2\omega t(\cos 2\omega t + 2\cos 4\omega t)] \tag{17f}$$

$$g_7 = -g_4 + (\lambda^2\delta/32\omega^4)(1 - 4\cos 2\omega t + 3\cos 4\omega t + 4\omega t\sin 4\omega t) \tag{17g}$$

$$g_8 = (\lambda/2\delta)g_6 + (\lambda^2/32\omega^2)(4\sin 2\omega t - 3\sin 4\omega t + 4\omega t\cos 4\omega t). \tag{17h}$$

In the application of SCS to the problem of free-electron lasers, we typically have $N \gg 1$. Therefore equation (12) is a very accurate approximation to a large number of leading terms in the series of equation (10). Then we have a very neat asymptotic expression for the probability distribution

$$Q(\theta, \phi, t) \cong [F^2(\theta, \phi, t) + 2N\varepsilon G(\theta, \phi, t)]^{N/2}. \tag{18}$$

This final result can be considered as an almost exact solution to the spherical RN equation.

The detailed derivation and the implications of this solution will be published elsewhere.

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